Unit - V

Absolutely continuous function

Definition:

A real valued function f on a closed bounded interval $[a,b]$ is said to be absolutely continuous on $[a,b]$ provided for each $\varepsilon>0$ there is a $\delta>0$ such that for every finite disjoint collection $\{(a_k,b_k)\}_{k=1}^{n}$ of open intervals in (a,b) If $\sum_{k=1}^{n}[b_k-a_k]<\delta$ Then $\sum_{k=1}^{n}|f(b_k)-f(a_k)|<\varepsilon$

Note:

If we take only one interval (x,y) in (a,b) instead of a finite disjoint collection then $|x-y|<\delta \Rightarrow |f(x)-f(y)|<\varepsilon$

In this sense every absolutely function on $[a,b]$ is uniformly continuous and hence f is continuous but the converse is not true.

Result:

Linear combination of absolutely continuous function are absolutely continuous.

proof: Let f and g be two real valued absolutely continuous functions defined on a closed bounded interval $[a,b]$

Let $\varepsilon>0$ be given

Let $\{(a_k,b_k)\}_{k=1}^{n}$ be a finite disjoint collection of open intervals in (a,b).

Since f is absolutely continuous, $\delta>0$ such that if $\sum_{k=1}^{n}|(b_k-a_k)|<\delta$ Then $\sum_{k=1}^{n}|f(b_k)-f(a_k)|<\varepsilon/2$

Since g is absolutely continuous, $\exists$ $\delta>0$ such that if $\sum_{k=1}^{n} |(b_k - a_k)| < \delta$ then $\sum_{k=1}^{n} |g(b_k) - g(a_k)| < \epsilon/2$

$$\sum_{k=1}^{n} |(f+g)(b_k) - (f+g)(a_k)| = \sum_{k=1}^{n} |f(b_k) - f(a_k) + g(b_k) - g(a_k)|$$

$$\leq \sum_{k=1}^{n} |f(b_k) - f(a_k)| + \sum_{k=1}^{n} |g(b_k) - g(a_k)|$$

$$< \epsilon/2 + \epsilon/2 = \epsilon$$

$\therefore$ $f+g$ is absolutely continuous

Suppose α be a real number

Now, for given $\epsilon > 0$, $\exists\ \delta > 0$ if $\sum_{i=1}^{n} |(b_k - a_k)| < \delta$ then $\sum_{i=1}^{n} |f(b_k) - f(a_k)| < \frac{\epsilon}{|\alpha|}$

$$\sum_{i=1}^{n} |(\alpha f)(b_k) - (\alpha f)(a_k)| = |\alpha| \sum_{k=1}^{n} |f(b_k) - f(a_k)|$$

$$< |\alpha| \frac{\epsilon}{|\alpha|}$$

$$\leq \epsilon$$

$\therefore$ αf is absolutely continuous

$\therefore$ Linear combination of absolutely continuous functions are absolutely continuous.

Note: 1

Composition of two absolutely continuous functions need not be absolutely continuous.

For, consider $f(x) = x^{1/3}$, $-1 \leq x \leq 1$

$$g(x) = \begin{cases} x^2 (\cos(\pi/2)) & ; x \neq 0, x \in [-1,1] \\ 0 & ; x = 0 \end{cases}$$

Then f and g are absolutely continuous function

But $f \circ g$ is not of bounded variation and hence $f \circ g$ is not absolutely continuous.

proposition:

If the function is lipschitz on a closed bounded interval $[a, b]$. Then f is the absolutely continuous on $[a,b]$.

proof: If the function f is lipschitz constant for the function f on $[a, b]$

Then $|f(u) - f(v)| \leq c|u-v|$ for all $u, v \in [a,b]$

Let $\varepsilon > 0$ be given

Let $\{(a_k, b_k)\}_{k=1}^{n}$ be a finite desjoint collection of open intervals in (a, b).

Then $|f(b_k) - f(a_k)| \leq c|b_k - a_k|$ $\forall$ $k = 1, 2 \ldots n$ — ①

Suppose there exit $\delta > 0$ such that $\sum_{k=1}^{n} (b_k - a_k) < \delta$

Then $(b_k - a_k) < \delta$ $\forall$ $k = 1, 2, \ldots n$

$|f(b_k) - f(a_k)| \leq n$

If we choose δ. small that $n\delta < \varepsilon/c$

then if $\sum_{k=1}^{n} (b_k - a_k) < \delta$

Then $\sum_{k=1}^{n} |f(b_k) - f(a_k)| < \varepsilon$

$\therefore$ f is absolutely continuous on $[a, b]$.

Note:

$f : [0,1] \rightarrow R$ defined by $f(x) = \sqrt{x}$ is absolutely continuous on $[0,1]$ but f is not lipschitz on $[0,1]$

Theorem:

Let the function f be absolutely continuous on the closed bounded interval $[a,b]$. Then f is the difference of two increasing absolutely continuous function and in particular is of bounded variation.

proof:

First we prove f is a function of bounded variation on $[a,b]$

To prove: $T_V(f) < \infty$

Given f is absolutely continuous

Let $\epsilon = 1$

Then there exist $\delta > 0$ such that for every finite disjoint collection $\{(a_k, b_k)\}_{k=1}^{n}$ of open intervals in (a,b) if $\sum_{k=1}^{n} (b_k - a_k) < \delta$ then $\sum_{k=1}^{n} |f(b_k) - f(a_k)| < 1$

$$\Rightarrow |f(b_k) - f(a_k)| < 1 \quad \forall k = 1 \cdots n$$

Let P be a partition of $[a,b]$ into N, closed interval $\{[c_k, d_k]\}_{k=1}^{n}$ each of length $< \delta$.

Then by the absolute continuity of f w.r. to the response f for the challenge $\epsilon = 1$

$$T_V(f[c_k, d_k]) \leq 1 \quad \text{for } 1 \leq k \leq N$$

$$T_V(f) = \sum_{k=1}^{N} T_V(f[c_k, d_k]) \leq N < \infty$$

$\therefore$ f is bounded variation we can express as

$$f(x) = [f(x) + T_V(f[a,x])] - T_V(f[a,x])$$

where $f(x) + T_V(f[a,x])$ and $T_V(f[a,x])$ are incr. function

To prove the theorem, it is enough to prove that $T_V(f[a,x])$ is absolutely continuous for all $x \in [a,b]$

Let $\{(c_k, d_k)\}_{k=1}^{n}$ be a disjoint collection of open sub interval of (a,b) Then there exist $\delta > 0$ such that if $\sum_{k=1}^{n} (d_k - c_k) < \delta$ then $\sum_{k=1}^{n} |f(d_k) - f(c_k)| < \epsilon/2$

For $1 \leq k \leq n$. Let P_k be a partition on $[c_k, d_k]$

By the choice of δ in relation in the absolutely continuity of f on $[a,b]$.

$$\sum_{k=1}^{n} V(f[c_k, d_k], P_k) < \varepsilon/2$$

Taking the sup as far $1 \leq k \leq n$ where P_k vary among partitions of $[c_k, d_k]$

we obtain $\sum_{k=1}^{n} T_V(f[c_k, d_k]) < \varepsilon/2 < \varepsilon$ —①

W.K.T for $1 \leq k \leq n$,

$T_V(f[c_k, d_k]) = T_V(f[a, d_k] - T_V(f[a, c_k])$ —②

If $\sum_{k=1}^{n} (d_k - c_k) < \delta$ Then $\sum_{k=1}^{n} [T_V(f[a, d_k] - T_V(f(a, c_k)] < \varepsilon$

This show that $T_V(f)$ is absolutely continuous $[a,b]$

$\therefore$ $f(x) + T_V(f[a,x])$ and $T_V(f[a,x])$ are absolutely continuity functions on $[a,b]$

Result:

Linear combination of absolutely continuous is absolutely.

Defenition:

A family f of measurable functions on E is said to be uniformly integrable over E provided for each $\varepsilon > 0$ if $\delta > 0$ such that for each $f \in F$ if $A \subseteq E$ is measurable and $m(A) < \delta$ then $\int_A |f| < \varepsilon$.

Theorem : 9

Let the function f be continuous on the closed bounded interval $[a, b]$. Then f is absolutely continuous on $[a, b]$ iff the family of divided difference function $\{Diff_h f\}_{0<h<1}$ is uniformly integrable over $[a, b]$

Proof :-

Assume $\{Diff_h f\}_{0<h\leq 1}$ is uniformly integrable over $[a, b]$

Let $\epsilon > 0$, then $\delta > 0$ for which $\int_E |Diff_h f| < \epsilon/2$ if $m(E) < \delta$ and $0 < h \leq 1 \rightarrow$ ①

Let $\{(c_k, d_k)\}_{k=1}^{n}$ be a disjoint collection of open subintervals of (a, b) for which $\sum_{k=1}^{n} (d_k - c_k) < \delta$

W.K.T f_h, for $0 < h \leq 1$ & $1 \leq k \leq n$

$$\int_{c_k}^{d_k} Diff_h f = AV_h f(d_k) - AV_h f(c_k)$$

$$\sum_{k=1}^{n} |(AV_h f(d_k) - AV_h f(c_k))| = \sum_{k=1}^{n} \left| \int_{c_k}^{d_k} Diff_h f \right|$$

$$\sum_{k=1}^{n} |AV_h f(d_k) - AV_h f(c_k)| \leq \sum_{k=1}^{n} \int_{c_k}^{d_k} |Diff_h f|$$

$$= \int_E |Diff_h f|$$

where $E = \bigcup_{k=1}^{n} (c_k, d_k)$ and $m(E) < \delta$

By the choice of δ, $\sum_{k=1}^{n} |Av_h f(d_k) - Av_h f(c_k)| < \frac{\epsilon_1}{2}$

This is true for all $0 < h \leq 1$

Given that f is continuous

$\therefore$ Taking the limit as $h \to 0^+$ we obtain

$\sum_{k=1} |f(d_k) - f(c_k)| \leq \epsilon_1/2 < \epsilon_1$

$\therefore$ f is absolutely continuous

Necessary part

Conversely, suppose f is absolutely continuous

W.K.T, f is difference of two increasing abso-lutely continuous function

We can assume that f is increasing

Then the divided difference functions are non-negative we have to prove $\{Diff_h f\}_{0<h\leq 1}$ is uniformly integrable over $[a,b]$.

Let $\epsilon_1 > 0$ we have to find a $\delta > 0$ such that for each measurable subset E of (a,b)

$\int_E Diff_h f < \epsilon_1$ if $m(E) < \delta$ and $0 < h \leq 1 \rightarrow (*)$

W.K.T if E is measurable set then there exist G_δ set G for which $m(G \sim E) = 0$

But every G_δ set is the intersection of a descending sequence of open set and every open set in the disjoint union of countable collection of open intervals

(ie) Every open set is a union of an ascending sequence of open sets, each of which is the union of a finite disjoint collection of open intervals

$\therefore$ To prove $\circledast$ it is enough to find $\delta > 0$ such that for $\{(c_k, d_k)\}_{k=1}^{n}$ a disjoint collection of open intervals in (a,b)

$\int_E \text{Diff}_h f < \epsilon/2$ if $m(E) < \delta$ where $E = \bigcup_{k=1}^{n} (c_k, d_k)$

and $0 < h \leq 1$

Choose $\delta > 0$ as the responce to the $\frac{\epsilon}{2}$ challenge regrading critorian for the absolutely continuinity of f on $[a, b+1]$

Let $a \leq u < v \leq b$

Then $\int_u^v \text{Diff}_h f = \frac{1}{h} \int_0^h g(t)dt$ where $g(t) = f(v+t) - f(u+t)$

For $0 \le t \le 1 \rightarrow (1)$

$$\int_u^v \mathrm{Diff}_h f = \int_u^v \frac{f(x+h) - f(x)}{h} dx$$

$$= \frac{1}{h}\left[\int_u^v f(x+h)\,dx - \int_u^v f(x)\,dx\right]$$

$$= \frac{1}{h}\left[\int_{u+h}^{v+h} f(t)\,dx - \int_u^v f(x)\,dx\right]$$

$$= \frac{1}{h}\left[\int_0^h f(v+t)\,dt - \int_0^h f(u+t)\,dt\right]$$

$$= \frac{1}{h}\int_0^h [f(v+t) - f(u+t)]\,dt$$

$$= \frac{1}{h}\int_0^h g(t)\,dt$$

If $\{(c_k, d_k)\}_{k=1}^n$ is a disjoint collection of open subintervals (a,b)

$$\int_E \mathrm{Diff}_h f = \frac{1}{h}\int_0^h g(t)\,dt \quad \text{where } E = \bigcup_{k=1}^n (c_k, d_k)$$

and $g(t) = \sum_{k=1}^n [f(d_k+t) - f(c_k+t)] \;\forall\; 0 \le t \le 1$

If $\sum_{k=1}^n (d_k - c_k) < \delta$ then For $0 \le t \le 1$

$\sum_{k=1}^n [f(d_k+t) - f(c_k+t)] < \epsilon/2$ and $g(t) < \epsilon/2$

Thus $\int_E \mathrm{Diff}_h f = \frac{1}{h}\int_0^h g(t)\,dt < \epsilon/2$

$$= \frac{1}{h}\,\epsilon/2 \int_0^h dt = 1/h \times \epsilon/2 \times h = \epsilon/2$$

Discrete Formulation of Fundamental thm for integral calculus

Defn:-

Let f be a continuous function on a closed bounded interval $[a,b]$. Then let $\int_a^b Diff_h f = Av_h f(b) - Av_h f(a)$

Note:-

Since f is continuous

$\lim_{h \to 0^+} Av_h f(b) - Av_h f(a) = f(b) - f(a)$

$\therefore \lim_{h \to 0^+} \int_a^b Diff_h f = f(b) - f(a)$

Theorem: Fundamental thm of integral calculus for Lebesgue Integral

Let the function f be absolutely continuous on the closed bounded interval $[a,b]$. Then f is differentiable almost everywhere on (a,b) its derivatives f' is integrable over $[a,b]$ and

$\int_a^b f' = f(b) - f(a)$

proof :-

W.K.T $\lim_{h \to 0^+} \int_a Diff_h f - f(b) - f(a)$

Put $h = 1/n$

Then $\lim_{n\to\infty} \int_0^b \text{Diff}_{1/n} f = f(b) - f(a) \longrightarrow (1)$

Given f is absolutely continuous on $[a,b]$

$\therefore f$ is difference of two increasing function on $[a,b]$

But increasing functions are differentiable almost everywhere

(If the function f is monotonic on the interval (a,b) then it is differentiable almost everywhere on (a,b))

$\therefore f$ is differentiable almost everywhere on (a,b)

i.e.) $\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$ exist almost everywhere and equal to f'

i.e.) $\lim_{h\to 0} \text{Diff}_h f$ exists almost everywhere is equal to f'

i.e.) $\lim_{n\to\infty} \text{Diff}_{1/n} f$ exist almost everywhere and is equal to f'

i.e., $\{Diff_{1/n} f\}$ converges pointwise almost every-where on (a, b) to f'

On the other hand the family $\{Diff_{1/n} f\}$ is uniformly integrable over $[a, b]$

By vitali converges thm

$$\lim_{n\to\infty} \int_a^b Diff_{1/n} f = \int_a^b \lim_{n\to\infty} Diff_{1/n} f = \int_a^b f' \rightarrow (2)$$

From (1) & (2) we have

$$f(b) - f(a) = \int_a^b f'$$

Defn:-

A function f on a closed, bounded inter-val $[a, b]$ the indefinite integrable of g over $[a, b]$ provided g is lebesgue integrable over $[a, b]$ and $f(x) = f(a) + \int_a^x g$ for all $x \in [a, b]$.

Theorem : 11

A function f on a closed bounded interval $[a,b]$ is absolutely continuous on $[a,b]$ iff it is an indefinite integral over $[a,b]$

Pf

Suppose f is absolutely continuous on $[a,b]$

Then f is absolutely continuous on $[a,x]$ $\forall x \in [a,b]$

In the theorem replacing $[a,b]$ by $[a,x]$, we have

$$\int_a^x f' = f(x) - f(a)$$

$$\therefore f(x) = f(a) + \int_a^x f'$$

This shows that f is indefinite integral of f' over $[a,b]$

Conversely,

Suppose that f is the indefinite integral of g over $[a,b]$

Then $f(x) = f(a) + \int_a^x g, \ \forall x \in [a,b] \rightarrow (1)$

Let $\{(a_k, b_k)\}_{k=1}^{n}$ be a finite disjoint collection of open subintervals of (a,b)

Let $E = \bigcup_{k=1}^{n} (a_k, b_k)$

$$(1) \Rightarrow \sum_{k=1}^{n} |f(b_k) - f(a_k)| = \sum_{k=1}^{n} \left| \int_{a_k}^{b_k} g \right|$$

$$\leq \sum_{k=1}^{n} \int_{a_k}^{b_k} |g|$$

$$= \int_E |g|$$

Let $\varepsilon > 0$

Since g is integrable, $\exists\ \delta > 0$ suchthat $\int_E |g| < \varepsilon$ if $E \subseteq [a,b]$ is measurable and $m(E) < \delta$

$\therefore$ If $\sum_{k=1}^{n} (b_k - a_k) < \delta$, then $\int_E |g| < \varepsilon$

Since $E = \bigcup_{k=1}^{n} (a_k, b_k)$ ie) if $\sum_{k=1}^{n} (b_k - a_k) < \delta$ then $\sum_{k=1}^{n} |f(b_k) - f(a_k)| < \varepsilon$

$\therefore f$ is absolutely Continuous.

<u>Corollary: 12</u>

Let the function f be monotonic on the closed bounded interval $[a,b]$. Then f is absolutely Continuous on $[a,b]$ iff $\int_a^b f' = f(b) - f(a)$

Pf:

Given f is monotonic on the closed bounded interval on $[a,b]$

Suppose f is absolutely Continuous on $[a,b]$ Then By the fundamental Theorem of integral Calculus, $\int_a^b f' = f(b) - f(a)$

Conversely,

Suppose f is increasing and $\int_a^b f' = f(b) - f(a)$

Let $x \in [a,b]$

Now, $\int_a^b f' - (f(b) - f(a)) = 0$

ie) $\int_a^x f' - (f(x) - f(a)) + \int_x^b f' - (f(b) - f(x)) = 0$

By an earlier Corollary, $\int_a^b f' \leq f(b) - f(a)$

$\therefore \int_a^x f' - (f(x) - f(a)) \leq 0$ and $\int_x^b f' - (f(b) - f(x)) = 0$

First one implies $\int_a^x f' = f(x) - f(a)$

ie) $f(x) = f(a) + \int_a^x f'$

ie) f is the indefinite integral of f' over $[a,b]$

$\therefore$ By the above Theorem, f is absolutely continuous on $[a,b]$

Lemma : 13

Let f be integrable over the closed, bounded interval $[a,b]$. Then $f(x)=0$ for almost all $x \in [a,b]$ iff $\int_{x_1}^{x_2} f = 0$ for all $(x_1, x_2) \subseteq [a,b]$

Proof:

Suppose $f(x)=0$ for almost all $x \in [a,b]$

ie) $m(\{x \in [a,b] \mid f(x) \neq 0\}) = 0$

Now, $\int_{x_1}^{x_2} f = \int_A f + \int_B f$ where $A = \{x \in (x_1, x_2) \mid f(x) = 0\}$

$B = \{x \in (x_1, x_2) \mid f(x) \neq 0\}$

Clearly $m(B) = 0$

$\int_{x_1}^{x_2} f = \int_A 0 + \int_B f$ where $m(B) = 0$

$\int_{x_1}^{x_2} f = 0$

This is true for all $(x_1, x_2) \subseteq [a,b]$

Conversely,

Suppose $\int_{x_1}^{x_2} f = 0 \; \forall (x_1, x_2) \subseteq [a,b]$

We claim that, $\int_E f = 0 \; \forall$ measurable sets $E \subseteq [a,b]$

For,

we know that every openset is the union of a countable disjoint collection of open interval.

$\therefore$ If G is a openset contained in $[a,b]$ then $G = \bigcup_{k=1}^{\infty} G_k$ where G_k's are disjoint open interval in $[a,b]$

$\therefore \int_G f = \sum_{k=1}^{\infty} \int_{G_k} f = 0 \rightarrow ①$ $\left[\because \int_{G_k} f = 0 \; \forall \; k = 1 \text{ to } \infty\right]$

Since G_δ is a countable intersection of a decending collection of opensets.

By the continuity of integration, $\int_{G_\delta} f = 0$

Now, Every measurable subset of $[a,b]$ is of this form $G \sim E_0$ where G is a G_δ set and $m(E_0) = 0$

[(ie) If E is measurable subset of $[a,b]$, then $\exists$ a G_δ set such that $E \subseteq G$ and $m(G \sim E) = 0$]

Put $E_0 = G \sim E$

Then $G \sim E_0 = E$

$\int_E f = 0 \rightarrow (*)$ for every measurable subset E of $[a,b]$

Hence our claim.

Let $E^+ = \{x \in [a,b] \mid f(x) \geq 0\}$ and

$E^- = \{x \in [a,b] \mid f(x) \leq 0\}$

There are measurable subsets of $[a,b]$

$$\therefore \int_a^b f^+ = \int_{E^+} f = 0 \ [\text{by} \circledast] \ [\because f^+ = \max\{f(x), 0\}]$$

Similarly, $$\int_a^b -f^- = -\int_{E} f = 0 \ [\text{by} \circledast] \ [\because f^- = \max\{-f(x), 0\}]$$

Since f^+ and f^- are non-negative, $\int_a^b f^+ = 0$ and $\int_a^b f^- = 0$

$\Rightarrow f^+$ and f^- are zero.

Since $f = f^+ - f^-$, $f = 0$ almost everywhere on $[a,b]$

Theorem : 14

Let f be integrable over the closed, bounded interval $[a,b]$. Then $\frac{d}{dx}\left[\int_a^x f\right] = f(x)$ for almost all $x \in (a,b)$

Proof:

Define F on $[a,b]$ by $F(x) = \int_a^x f \ \forall x \in [a,b]$

Since F is an indefinite integral, F is also a absolutely continuous $\left[\because F(a) = \int_a^a f = 0, \ f(x) = f(a) + \int_a^x f\right]$

$\therefore F$ is differentiable almost everywhere on (a,b)

It's derivative F' is integrable and

$$\int_a^b F' = F(b) - F(a).$$

We have to prove $\frac{d}{dx}\left[\int_a^x f\right] = f(x)$ for almost all $x \in (a,b)$

ie) TP $\frac{d}{dx} F(x) = f(x)$ almost everywhere (ie) TP $F' - f = 0$ almost everywhere

By the above lemma,

It is enough to prove $\int_{x_1}^{x_2} (F'-f) = 0 \ \forall (x_1, x_2) \subseteq [a,b]$

Now, $\int_{x_1}^{x_2} (F'-f) = \int_{x_1}^{x_2} F' - \int_{x_1}^{x_2} f \rightarrow (1)$

But $\int_{x_1}^{x_2} F' = F(x_2) - F(x_1)$ [$\because$ F is absolutely continuous]

$$= \int_a^{x_2} f - \int_a^{x_1} f$$

In (1) $\Rightarrow \int_{x_1}^{x_2} (F'-f) = \int_a^{x_2} f - \int_a^{x_1} f - \int_{x_1}^{x_2} f$

$$= \int_a^{x_2} f - \left(\int_a^{x_1} f + \int_{x_1}^{x_2} f \right)$$

$$= \int_a^{x_2} f - \int_a^{x_2} f$$

$$\int_{x_1}^{x_2} (F'-f) = 0$$

$\therefore F' - f = 0$ almost everywhere

$\frac{d}{dx}[F(x)] = f(x)$ almost everywhere.

Hence $\frac{d}{dx}\left[\int_a^x f\right] = f(x)$ for almost all $x \in (a,b)$

Definition:-

A function of bounded variation is said to be singular provided its derivative vanishes almost everywhere.

Example:

1. The Cantor Lebesgue function is a non-constant singular function.

2. An absolutely continuous function is singular iff it is a constant.

Definition:

Let f be of bounded variation on $[a,b]$. Then f' is integrable over $[a,b]$. Define $g(x) = \int_a^x f'$ and

$h(x) = f(x) - \int_a^x f'$ $\forall$ $x \in [a,b]$ such that $f = g + h$ on $[a,b]$

g is absolutely continuous and h is singular. This decomposition of f is called Lebesgue decomposition of f.

Section - 17.1.

By a measurable space we mean a couple (X, M) consisting subset of X and a σ-algebra M of subsets of X.

Note:

A subset E of X is said to be measurable provided $E \subset M$.

A measure μ on a measurable space (X, M) is a function $\mu: M \to [0, \infty]$ with the following properties $\mu(\phi) = 0$.

For any countable collection $\{E_k\}_{k=1}^{\infty}$ of disjoint measurable subset of X. $\mu\left(\sum_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu(E_k)$

A triple (X, M, μ) is said to be measure space provided (X, M) is a measurable space and μ is measure on M.

Example:

(R, L, m) where R is the set of real numbers L is the set of all Lebesgue measurable sets m is the Lebesgue measure is measurable space.

Definition:

Let (X, M, μ) be a measurable space.

i) The measure μ is said to be finite provided $\mu(X) < \infty$

ii) The measure μ is said to be σ-finite provided $X = \bigcup_{i=1}^{\infty} X_i, \ldots X$ are measurable sets with $\mu(X_i) < \infty$ for each i.

iii) A measurable set E is said to be of finite measure provided $\mu(E) < \infty$.

iv) A measurable set E is said to be of σ-finite measure provided $E = \bigcup_{i=1}^{\infty} E_i$, $E_i \subset M$ with $\mu(E_i) < \infty$ for each i.

* A measure space (X, M, μ) is said to be complete provided every subset of sets of measurable zero is measurable.

(i.e) If $A \in M$ with $\mu(A) = 0$ and $B \subseteq A$ then $B \in M$.

Example: (R, L, m) is a complete measure space.

Preoposition-3

Every measure space has a complete. Let (X, M, μ) be a measure space. Define M_0 to be collection of subsets of M of the form $E = A \cup B$ where $B \in M$, $A \subseteq C$ for some $C \in M$ for which $\mu(C) = 0$

Proof:

Given that $M_0 = \{E = A \cup B \ ; \ B \in M, \ A \subseteq C \ ; \ C \in M, \ \mu(C) = 0\}$

To Prove: M_0 is a σ-algebra containing M.

Let $B \in M$.

Then $B = \phi \cup B \in M_0$.

$\therefore M \subseteq M_0$

Let $E \in M_0$.

(i.e) $E = A \cup B \ ; \ B \in M \quad A \subseteq C \ ; \ \mu(C) = 0$

$E^c = A_1 \cup B_1$, where $B_1 \in M$, $A_1 \subseteq C$ with $\mu(C_1) = 0$

Let $E = A \cup B \in M_0$.

Given that $\mu_0(E) = \mu(B)$.

T.P: (X, M_0, μ_0) is a complete measure space.

Let $E = A \cup B \in M_0$ with $\mu_0(E) = 0$

Let $E_1 \subseteq E$.

Then $E_1 = A_1 \cup B_1$ for some $A_1 \subseteq A$ for some $B_1 \subseteq B$.

Since $\mu_0(E) = \mu(B)$

$\therefore A_1 \cup B_1 \subseteq A \cup B$.

Also, $A \subseteq C$, $C \in M$, $\mu(B) = 0$; $B \in M$

$\therefore A_1 \cup B_1 \subseteq C \cup B$

Since $C \in M$, $B \in M$ $\quad \mu(C) = 0$; $\mu(B) = 0$.

Take $C_1 = B \cup C$, $A_2 = A_1 \cup B_1$.

Then $C_1 \in M$, $\mu(C_1) = 0$ & $A_2 \subseteq C_1$

Now, $E_1 = A_2 \cup \phi$

Since $\phi \in M$, $E_1 \in M_0$

$\therefore (X, M_0, \mu_0)$ is a complete measure

Section :- 17.2

Signed Measure

A signed measure ν on a measurable space (X, M) we mean a function $\nu : M \to [-\infty, \infty]$ which posses the following properties.

i) ν assumes almost one of $+\infty, -\infty$

ii) $\nu(\phi) = 0$

iii) For any countable $\{E_k\}_{k=1}^{\infty}$ disjoint collection of measurable sets $\mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu(E_k)$ when the series $\sum_{k=1}^{\infty} \nu(E_k)$ converges absolutely if $\nu\left(\bigcup_{k=1}^{\infty} E_k\right)$ is finite.

Definition:-

Let ν be a signed measure on a measurable space (X, M). A set A is said to be (+ve) with respect to ν provided

i) A is a measurable set $A \in M$

ii) For each measurable subset E of A, $\nu(E) \geq 0$

Remark:

Let A be a (+ve) set with respect to the signed measurable set ν. Then the restriction of ν of the measurable subsets of A is measure.

Definition:

Let ν be a signed measure on a measurable space (X, M). A set B is said to be (-ve) with respect to ν Provided, i) $B \in M$.

ii) For each measurable Subset E of B, $\nu(E) = 0$

Remark: The restriction of ν on the measurable subsets of (-ve) set B is measure.

Let γ be a signed measure on the measurable space (X, M). A measure set E is said to null set provided for each measurable subset A of E, $\gamma(A) = 0$.

Notes:

1. A measurable set E measure zero need not be a null set. Let $E = A \cup B$, $A, B \in M$. $\gamma(A) = -2$; $\gamma(A) = 2$.

Then $\gamma(E) = \gamma(A \cup B) \Rightarrow \gamma(A) + \gamma(B) = -2+2 = 0$

$\gamma(E) = 0$.

But $A \in M$, $A \subseteq E$, $\gamma(A) = -2 \neq 0$

$\therefore$ E is not all a null set.

2. ϕ is a null set

3. A set E is null with respect to a μ iff E has measure zero.

Proof:

Suppose E is a null set.

Each measurable subset has measure zero.

Since $E_1 \subset E$, $\mu(E) = 0$

Since μ is measurable, $\mu(E_1) \leq \mu(E) = 0$ for all measurable subset E_1 of E.

$\therefore \mu(E_1) = 0$.

Hence E is a null set with respect to measure μ.

4. If E is a null set with respect to the signed measure γ, then $\gamma(E) = 0$

Preposition - 4

Let γ be a signed measure on a measurable space (X, μ) then, i) Every measurable subset of a (+ve) set is (+ve)

ii) The union of countable collection of (+ve) set is also (+ve)

Proof:

i) Suppose A is (+ve) with respect to γ.

Let $E \subset A$, a measurable set.

Then by defn. $\gamma(E) \geq 0$.

Let E_1 be a measurable subset of E.

Then E_1 is a measurable subset of A also $\gamma(E_1) \geq 0$

E_1 is positive.

Every measurable, subset of a (+ve) set is (+ve)

ii) Let $A = \bigcup_{k=1}^{n} A_k$, A_k is a (+ve) set w.r.to γ.

Let E be a measurable subset of A

T.P: $\gamma(E) \geq 0$

Consider $E_1 = E \cap A_1$

Then E_1 is a measurable subset of A_1.

$\therefore \gamma(E_1) \geq 0$

Consider $E_1 = E \cap A_1$.

$\therefore E_2$ is measurable subset of A_2 and $\gamma(E_2) \geq 0$.

Choose $E_n = (E \cap A_n) \sim (A_1 \cup A_2 \cdots A_{n-1})$ $n = 1, 2 \ldots$

Then E_n is measurable and $\gamma(E_n) \geq 0$ $\forall n = 1, 2, \ldots$

$\therefore E = \bigcup_{k=1}^{\infty} E_k$.

E_k's are disjoint measurable sets $\gamma(E_k) \geq 0$.

$$\therefore \gamma(E) = \sum_{k=1}^{\infty} \gamma(E_k) \geq 0$$

A is a (+ve) set w.r. to γ.

Hahn's Lemma:

Let γ be a signed measure on a measurable space (X, M) & E is measurable set for which $0 < \gamma(E) < \infty$. Then there is a measurable subset A of E that is (+ve) and of +ve measure.

Proof:

If E is a (+ve) measurable set.

Then the thm. is proved.

Suppose E is not (+ve).

Then there is a measurable subset A of E for which $\gamma(A) < 0$.

Let m_1 be the smallest the (+ve) integer suchthat there is measurable subset A_1 of E suchthat $\gamma(A_1) < -1/m_1$.

For any n choose the smallest (+ve) integer $m_1, m_2, \ldots m_n$ and measurable subsets $A_1, A_2, \ldots A_n$, of E such that

$A_k \subset E \sim \left(\bigcup_{i=1}^{k-1} A_i\right)$; $\gamma(A_k) < -1/m_k$; $1 \leq k \leq n$.

Suppose the process of selecting A_k's terminate at the $(n-1)^{th}$ stage.

Take $E_1 = E \sim \left(\bigcup_{i=1}^{n-1} A_i\right)$

$$\gamma(E) = \gamma(E_1) + \sum_{i=1}^{n-1} \gamma(A_i)$$

Since $\gamma(E) > 0$, $\sum_{i=1}^{n-1} \gamma(A_i) < 0$, $\gamma(E_1) > 0$.

$\therefore E_1$ is a (+ve) set of (+ve) measure.

Otherwise we get a countable collection $\{A_k\}_{k=1}^{\infty}$ of measurable sets suchthat $\gamma(A_k) < -1/m_k$ ——— (1)

Take $A = E \sim \left(\bigcup_{k=1}^{\infty} A_k\right)$

Then A is a measurable subset of E

T.P: A is positive.

Let E_1 be a measurable subset of A.

$\therefore E_1 \subset A \subset E \sim \left(\bigcup_{i=1}^{k-1} A_i\right) \quad \forall i.$

$\gamma(E_1) \geq -1/m_k.$

Since $m_k \to \infty$ as $k \to \infty$, $\gamma(E_1) \geq 0.$

A is a (+ve) set.

Now, $E = A \cup \left(\bigcup_{k=1}^{\infty} \gamma(A_k)\right)$

$$\gamma(E) = \gamma(A) + \sum_{k=1}^{\infty} \gamma(A_k)$$

Since $\gamma(E) > 0$ and $\sum_{k=1}^{\infty} \gamma(A_k) < 0$, $\gamma(A) > 0$

$\therefore A$ is a set of (+ve) measure

Hahn Decomposition Theorem:

Let γ be a signed measure on a measurable space (X, M), then there is a (+ve) set A & (-ve) set B of γ for which $X = A \cup B$, $A \cap B = \phi$.

Proof:

Assumethat $+\infty$ is omitted by γ.

Let P be the collection of all (+ve) sets with respect to γ.

Consider $\lambda = \sup\{\gamma(A) \,;\, A \in P\}$

$\therefore$ There is a sequence $\{A_k\}_{k=1}^{\infty}$ of (+ve) sets suchthat $\gamma(A_k) \to \lambda$.

Take $A = \bigcup_{k=1}^{\infty} A_k.$

Then A is +ve set with respect to γ. & $\gamma(A) \leq \lambda$ ⊛

Now, $A = (A \sim A_k) \cup A_k$ is a measurable set.

$\gamma(A \sim A_k) \geq 0.$

Now, $A = (A \sim A_k) \cup A_k$.

$\gamma(A) = \gamma(A \sim A_k) + \gamma(A_k)$

$\gamma(A) \geq \gamma(A_k)$

$\therefore \gamma(A) \geq \lim\limits_{k \to \infty} \gamma(A_k)$

(i.e) $\gamma(A) \geq \lambda$ ——— $(**)$

From $(*)$ & $(**)$, hence $\gamma(A) = \lambda$ where A is (+ve).

Consider $B = X \sim A$.

To Prove: B is a (-ve) set.

It is clear that B is a measurable set.

Suppose B is (+ve) set.

Then there is a measurable set E of B for which $\gamma(E) > 0$. (i.e) $0 < \gamma(E) < \infty$.

$\therefore$ By Hahn's lemma, there is a (+ve) set $E_0 \subset E$, $\gamma(E_0) > 0$.

Now, $A \cup E_0$ is a (+ve) set.

Since $A \cap E_0 = 0$, $\gamma(A \cup E_0) = \gamma(A) + \gamma(E_0)$

$\gamma(A \cup E_0) > \lambda + \gamma(E_0)$

$\gamma(A \cup E_0) > \lambda$ which is a contradiction to $A \cup E_0$ is a (+ve) set.

$\therefore$ B is a (-ve) set.

Hence A is a (+ve) set B is a (-ve) set, $X = A \cup B$; $A \cap B = \phi$

Notes: 1) In the above theorem $\{A, B\}$ is called Hahn decomposition of X for γ.

2) If $\{A, B\}$ is a hahn decomposition of X for γ then $\{A \sim N, B \cup N\}$ is also a hahn decomposition of X for γ where N is a Null set. Hence Hahn decomposition is not unique.

Definition:

The two signed measure γ_1, γ_2 defined on a measurable set (X, M) are said to be mutually singular written $\gamma_1 \perp \gamma_2$ Provided there are measurable subset A, B of X for which

$X = A \cup B$, $A \cap B = \phi$; $\gamma_1(A) = 0$; $\gamma_2(B) = 0$

Example:

Let γ be a signed measurable space (X, M). Let $\{A, B\}$ be a hahn decomposition of X for γ.

Two measure γ^+, γ^- as follows. For any measurable set E

$\gamma^+(E) = \gamma(E \cap A)$; $\gamma^-(E) = -\gamma(E \cap B)$

Now,

$$E = (E \cap A) \cup (E \cap B)$$

$$\gamma(E) = \gamma(E \cap A) + \gamma(E \cap B)$$

$$= \gamma^+(E) - \gamma^+(E)$$

$$\gamma^+ \perp \gamma^-$$

$\gamma^+(B) = 0$; $\gamma(A) = 0$

Hence γ^+, γ^- are mutually singular.

Theorem-1

Let (X, M, μ) be a measure space (finite additionally)

i) For any finite disjoint collection $\{E_k\}_{k=1}^n$ of measurable sets $\mu\left(\bigcup_{k=1}^n E_k\right) = \sum_{k=1}^n \mu(E_k)$

ii) If A and B are measurable sets & $A \subset B$ then $\mu(A) \leq \mu(B)$.

iii) If moreover $A \subseteq B$ & $\mu(A) < \infty$ then $\mu(B - A) = \mu(B) - \mu(A)$ (Excision) so that If $\mu(A) = 0$ then $\mu(B \sim A) = \mu(B)$

iv) For any countable collection $\{E_k\}_{k=1}^\infty$ of measurable sets with covers a measurable set E,

$\mu(E) = \sum_{k=1}^\infty \mu(E_k)$ (countably monotonic)

Proof:

i) Finite additivity:

Since μ is countably additive, $\mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu(E_k)$ where $\{E_k\}_{k=1}^{\infty}$ is a countably collection of disjoint measurable.

Take $E_k = \phi \ \forall \ k > n$.

Then $\mu(E_k) = 0 \ \forall \ k > n$.

Now, $\bigcup_{k=1}^{\infty} E_k = \bigcup_{k=1}^{n} E_k$.

$$\mu\left(\bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} \mu(E_k)$$

ii) To Prove: monotonicity.

Given A and B are measurable sets $A \subseteq B$

Now, $B = A \cup (B \sim A)$

$\mu(B) = \mu(A) + \mu(B \sim A)$

$\mu(B) \geq \mu(A)$

iii) Exicision Property:

Let $A \subseteq B$ & $\mu(A) < \infty$.

To Prove: $\mu(B \sim A) = \mu(B) - \mu(A)$

$B = A \cup (B \sim A)$.

$\mu(B \sim A) = \mu(B) \sim \mu(A)$

If $\mu(A) = 0$ then $\mu(B \sim A) = \mu(B)$

iv) Countable monotonicity:

Given that $\{E_k\}_{k=1}^{\infty}$ is a measurable cover for E. (i.e) $E \subset \bigcup_{k=1}^{\infty} E_k$.

Define $G_1 = E_1$ $G_2 = E_2 \sim E_1$; $G_3 = E_3 \sim (E_1 \cup E_2)$

In general $G_k = E_k \sim \left(\bigcup_{i=1}^{k-1} E_i\right) \ \forall \ k \geq 2$, $\{G_k\}_{k=1}^{\infty}$ is a disjoint countable collection of measurable sets and $\bigcup_{k=1}^{\infty} G_k = \bigcup_{k=1}^{\infty} E_k$ & $G_k \subseteq E_k$

Since $E \subseteq \bigcup_{k=1}^{\infty} E_k = \bigcup_{k=1}^{\infty} G_k$.

By monotonicity, $\mu(E) \leq \mu\left(\bigcup_{k=1}^{\infty} G_k\right)$

$$\leq \sum_{k=1}^{\infty} \mu(G_k)$$

$$\mu(E) \leq \sum_{k=1}^{\infty} \mu(E_k)$$

Continuity of Measure:

Preposition-2:

Let (X, M, μ) be a measure space.

i) If $\{A_k\}_{k=1}^{\infty}$ is an ascending sequence of measurable sets then $\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \lim_{k\to\infty} \mu(A_k)$

ii) If $\{B_k\}_{k=1}^{\infty}$ is an decending sequence of measuarable sets which $\mu(B_1) < \infty$ then $\mu\left(\bigcap_{k=1}^{\infty} B_k\right) = \lim_{k\to\infty} \mu(B_k)$

Proof:

Case-i)

Given that $A_1 \subset A_2 \subset \cdots$

Suppose there is a k_0 such that $\mu(A_{k_0}) = \infty$

Then $\mu(A_k) = \infty \;\forall\; k \geq k_0$ & $\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \infty$

Hence $\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \infty = \lim_{k\to\infty} \mu(A_k)$

(i.e) $\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \lim_{k\to\infty} \mu(A_k)$

Case-ii)

Suppose $\mu(A_k) < \infty \;\forall\; k$.

Consider $A_0 = \phi$

Define $C_k = A_k \sim A_{k-1}$

Then $\{C_k\}_{k=1}^{\infty}$ is a disjoint collection of measurable sets and $\bigcup_{k=1}^{\infty} A_k = \bigcup_{k=1}^{\infty} A_k$

Now, $\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \mu\left(\bigcup_{k=1}^{\infty} C_k\right) = \sum_{k=1}^{\infty} \mu(C_k)$

$$\mu(C_k) = \mu(A_k \sim A_{k-1})$$

$$= \mu(A_k) \sim \mu(A_{k-1})$$

$$\therefore \sum_{k=1}^{n} \mu(C_k) = \sum_{k=1}^{n} [\mu(A_k) - \mu(A_{k-1})]$$

$$= [\mu(A_n) - \mu(A_{n-1})] + [\mu(A_{n-1}) + \mu(A_{n-2})] + \cdots + [\mu(A_2) - \mu(A_1)] + [\mu(A_1) - \mu(A_0)]$$

$$\sum_{k=1}^{n} \mu(C_k) = \mu(A_n) \quad \text{——— ①}$$

Hence $\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \lim_{n \to \infty} \left[\sum_{k=1}^{\infty} \mu(C_k)\right]$

$$= \lim_{n \to \infty} \mu(A_n)$$

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \lim_{k \to \infty} \mu(A_k)$$

ii) Given that $B_1 \supset B_2 \supset \cdots$; $\mu(B_1) < \infty$.

Consider $D_k = B_1 \sim B_k$.

$\therefore \{D_k\}$ is an ascending collection of measurable sets.

By i) $\mu\left(\bigcup_{k=1}^{\infty} D_k\right) = \lim_{k \to \infty} \mu(D_k)$

$$\bigcup_{k=1}^{\infty} D_k = \bigcup_{k=1}^{\infty} (B_1 \sim B_k)$$

$$= B_1 \sim \bigcap_{k=1}^{\infty} B_k$$

$$\mu\left(B_1 \sim \left(\bigcap_{k=1}^{\infty} B_k\right)\right) = \lim_{k\to\infty} \mu(D_k)$$

$$= \lim_{k\to\infty} \left[\mu(B_1 \sim B_k)\right]$$

$$\mu(B_1) - \mu\left(\bigcap_{k=1}^{\infty} B_k\right) = \lim_{k\to\infty} \left[\mu(B_1) - \mu(B_k)\right] \qquad [\because \text{By excision Property}]$$

$$= \mu(B_1) - \lim_{k\to\infty} \mu(B_k)$$

Since $\mu(B_1) < \infty$

$$\therefore \mu\left(\bigcap_{k=1}^{\infty} B_k\right) = \lim_{k\to\infty} \mu(B_k)$$

The Borel Contelli lemma:

Let (X, M, μ) be a measure space and $\{E_k\}_{k=1}^{\infty}$ Countable collection of measurable sets for which $\sum_{k=1}^{\infty} \mu(E_k) < \infty$. Then almost all x in X belong to almost finitely many of the E_k's.

Proof:

For each n, $\mu\left(\bigcup_{k=n}^{\infty} E_k\right) \leq \sum_{k=0}^{\infty} \mu(E_k) < \infty$.

Take $A_n = \bigcup_{k=n}^{\infty} E_k$.

Then $\{A_n\}_{n=1}^{\infty}$ is a decending collection of measurable sets with $\mu(A_1) < \infty$.

By continuity of measure thm,

$$\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n)$$

$$\mu\left(\bigcap_{n=1}^{\infty}\left(\bigcup_{k=n}^{\infty} E_k\right)\right) = \lim_{n\to\infty} \mu\left(\bigcup_{k=n}^{\infty} E_k\right)$$

$$\leq \lim_{n\to\infty} \sum_{k=n}^{\infty} \mu(E_k)$$

$$= 0$$

$$\therefore \mu\left(\bigcap_{n=1}^{\infty}\left(\bigcup_{k=n}^{\infty} E_k\right)\right) = 0$$

Hence almost all x in X belong to almost finitely many of the E_k's.

The Jordon decomposition thm:

Let ν be a Signed measure on a measurable space (X, M) then there are two mutually Singular measures ν^+, ν^- on (X, M) for which $\nu = \nu^+ - \nu^-$. More over there is only one such pair of mutually Singular measure.

Proof

Let $\{A, B\}$ be the Hahn decomposition of X for ν

Let E be any measurable set

Then $E = (E \cap A) \cup (E \cap B)$

$$\nu(E) = \nu(E \cap A) + \nu(E \cap B)$$

Define $\nu^+(E) = \nu(E \cap A)$

$$\nu^-(E) = -\nu(E \cap B)$$

Then ν^+, ν^- are measure on (X, M)

$$\nu(E) = \nu^+(E) - \nu^-(E)$$

$$\therefore \nu = \nu^+ - \nu^-$$

Claim: $\nu^+ \perp \nu^-$

Since $\nu^+(B) = 0 = \nu^-(A)$

$$\therefore \nu^+ \perp \nu^-$$

T.P: uniqueness of ν^+, ν^-

Suppose $\nu = \nu_1 - \nu_2$ where ν_1, ν_2 are two mutually singular measure on (X, M).

Then there are two measurable subset A_1, B_1 for which $X = A_1 \cup B_1$, $A_1 \cap B_1 = \phi$.

$$\nu_1(B_1) = 0 = \nu_2(A_1)$$

Claim: A_1 is a +ive set w.r. to ν

Let E be a measurable subset of A_1

Then $\nu(E) = \nu_1(E) - \nu_2(E)$

$= \nu_1(E) \quad [\because \nu_2(E) = 0]$

Since $\nu_1(E) > 0$, $\nu(E) > 0$

$\therefore A_1$ is a +ive set w.r to ν

$A - A_1$ is a null set say N

Now, $\gamma = \gamma^+ - \gamma^-$ and $\gamma = \gamma_1 - \gamma_2$

$$0 = (\gamma^+ - \gamma^-) - (\gamma_1 - \gamma_2)$$

$$(\gamma^+ - \gamma_1)(B) = \gamma^+(B) - \gamma_1(B) = 0$$

$$(\gamma^- - \gamma_2)(A) = \gamma^-(A) - \gamma_2(A) = 0$$

$$\therefore \gamma^+ - \gamma_1 \perp \gamma^- - \gamma_2$$

$$\gamma^- - \gamma_2 = 0$$

(ie) $\gamma_2 = \gamma^-$

Hence $\gamma_1 = \gamma^+$

$\therefore \gamma = \gamma^+ - \gamma^-$ is unique.

DEFN

i) $|\gamma|(E) = \gamma^+(E) + \gamma^-(E) \ \forall E \in M$

ii) $|\gamma|(X) = \sup \left\{ \sum_{k=1}^{n} |\gamma(E_k)| \right.$ where sup is taken over all finite disjoint collection of measurable set $|\gamma|(X)$ is called total variation of γ and is denoted by $\|\gamma\|$ variation

The carathedory measure induced by an outer measure

DEFN:

Let S be a collection of sets in X. A function $\mu : S \to [0, \infty]$ is said to be countably monotone provided whenever a sets $E \in S$ is covered by countable collection of sets $\{E_k\}$ in S, $\mu(E) \leq \sum_{k=1}^{\infty} \mu(E_k)$.

DEFN:

A countably monotone set $\mu : S \to [0, \infty]$ is said to be finitely monotone provided $\mu(\phi) = 0$

Note:

μ is finitely monotone is the sense that if $E \subset \bigcup_{k=1}^{n} E_k$ then $\mu(E) \leq \sum_{k=1}^{n} \mu(E_k)$

Proof:

If $E \subset \bigcup_{k=1}^{n} E_k$ then $E \subset \bigcup_{k=1}^{\infty} E_k$, $E_k = \phi \ \forall \ k > n$

$$\mu(E) \leq \sum_{k=1}^{\infty} \mu(E_k)$$

$$= \sum_{k=1}^{\infty} \mu(E_k) + \sum_{k=n+1}^{\infty} \mu(E_k)$$

$$= \sum_{k=1}^{n} \mu(E_k)$$

$$\mu(E) \leq \sum_{k=1}^{n} \mu(E_k)$$

Note:

If $A \subseteq B$, $A, B \in S$ then $\mu(A) \leq \mu(B)$

Finitely monotone set function is a monotone

DEFN:

A set function $\mu^* : 2^X \to [0, \infty]$ is said to be an outer measure provided

i) $\mu^*(\phi) = 0$

ii) μ^* is countably monotonicity.

DEFN:

For an outer measure $\mu^* : 2^X \to [0, \infty]$. A subset E of X is said to be measurable provided for any subset A of X,

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

Note:

Since μ^* is finitely monotone and

$$A = (A \cap E) \cup (A \cap E^c)$$

Always $\mu^*(A) \leq \mu^*(A \cap E) + \mu^*(A \cap E^c)$

Hence E is measurable w.r.to μ^* iff

$\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E^c)$

Remark:

Every set of measure zero is measurable.

Let $E = 2^*$ suchthat $\mu^*(E) = 0$

Let $A \in 2^*$ be any set

Then $A = (A \cap E) \cup (A \cap E^c)$

Since μ^* is finitely measure.

$\mu^*(A) \leq \mu^*(A \cap E) + \mu^*(A \cap E^c)$

Since μ^* is monotone $A \cap E \subset E$.

$\mu^*(A \cap E) \leq \mu^*(E) = 0$

$\mu^*(A \cap E) = 0$ —— (1)

$\mu^*(A) \leq \mu^*(A \cap E^c)$.

Since $A \cap E^c \subset A$, $\mu^*(A \cap E^c) \leq \mu^*(A)$

$\therefore \mu^*(A) \geq \mu^*(A \cap E^c)$

$= 0 + \mu^*(A \cap E^c)$

$\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E^c)$

$\therefore E$ is measurable w.r. to μ^*.

Proposition: 5

The union of a finite collection of measurable set is measurable.

Proof:

we first show that union of two measurable set is measurable.

Let E_1, E_2 be any two measurable sets.

Let A be any set in X.

$$\mu^*(A) \geq \mu^*(A \cap E_1) + \mu^*(A \cap E_1^c)$$

$$\geq \mu^*(A \cap E_1) + \{\mu^*(A \cap E_1^c) \cap E_2 + \mu^*(A \cap E_1^c) \cap E_2^c\}$$

$$= \mu^*(A \cap E_1) + \mu^*((A \cap E_1^c) \cap E_2) + \mu^*(A \cap (E_1 \cup E_2))$$

Claim: $\mu^*(A) \geq \mu^*(A \cap (E_1 \cup E_2)) + \mu^*(A \cap (E_1 \cup E_2)^c)$

Now: $A \cap [E_1 \cup E_2] = (A \cap E_1) \cup ((A \cap E_1^c) \cap E_2)$

$\mu^*(A \cap [E_1 \cup E_2]) \leq \mu^*(A \cap E_1) + \mu^*[A \cap (E_1 \cup E_2)^c]$

$\therefore E_1 \cup E_2$ is a measurable set.

next to prove finite union of measurable set is measurable by induction on n.

If E_1, E_2 be two measurable zero then $E_1 \cup E_2$ is measurable.

The result is true for $n = 2$.

Assume that the result have been proved upto $n-1$

(ii) $E_1, E_2, \ldots, E_{n-1}$ are measurable sets then $\bigcup_{i=1}^{n-1} E_i$ is measurable.

Suppose $E_1, E_2, \ldots, E_{n-1}, E_n$ are measurable sets then $\bigcup_{i=1}^{n} E_i$ is measurable

Since $\left(\bigcup_{i=1}^{n-1} E_i\right) \cup E_n = \bigcup_{i=1}^{n} E_i$

Hence the union of finite collection of measurable set is measurable.

Proposition : 6

Let $A \subseteq X$ and $\{E_k\}_{k=1}^{n}$ be a finite disjoint collection of measurable sets then $\mu^*\left(A \cap \left(\bigcup_{k=1}^{n} E_k\right)\right) = \sum_{k=1}^{n} \mu^*(A \cap E_k)$. In particular the restriction of μ^* to the collection of measurable set is finitely additive.

Proof:

we prove it by induction on n.

If $n=1$, then trivially $\mu^*(A \cap E_1) = \mu^*(A \cap E_1)$

(ie) The thm is true for $n=1$.

Assume that the thm has been proved upto $n-1$

(ie) $\mu^*\left(A \cap \left(\bigcup_{k=1}^{n-1} E_k\right)\right) = \sum_{k=1}^{n-1} \mu^*(A \cap E_k)$

Now prove the thm for n

now $\left(A \cap \left(\bigcup_{k=1}^{n} E_k\right)\right) \cap E_n = A \cap E_n$ and

$$\left(A \cap \left(\bigcup_{k=1}^{n} E_k\right)\right) \cap E_n^c = A \cap \left(\bigcup_{k=1}^{n-1} E_k\right)$$

Since E_n is measurable set

$$\mu^*\left(A \cap \left(\bigcup_{k=1}^{n} E_n\right)\right) = \mu^*\left(A \cap \left(\bigcup_{k=1}^{n} E_k\right) \cap E_n\right) + \mu^*\left(A_n \cap \left(\bigcup_{k=1}^{n} E_k\right) \cap E_n^c\right)$$

$$= \mu^*(A \cap E_n) + \mu^*\left(A \cap \left(\bigcup_{k=1}^{n-1} E_k\right)\right)$$

$$= \mu^*(A \cap E_n) + \sum_{k=1}^{n-1} \mu^*(A \cap E_k)$$

Hence by induction hypothesis

$\mu^*\left(A \cap \left(\bigcup_{k=1}^{n} E_k\right)\right) = \sum_{k=1}^{n} \mu^*(A \cap E_k)$ for any n.

Take $A = X$

Then $A \cap \left(\bigcup_{k=1}^{n} E_k\right) = \bigcup_{k=1}^{n} E_k$, $A \cap E_k = E_k$.

$$\mu^*\left(\bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} \mu^*(E_k)$$

$\therefore \mu^*$ is finitely subadditive

Proposition : 7

The union of a countable collection of measurable set is measurable.

Proof:

Since countable union of measurable sets can be wrotten as a countable union of disjoint measurable sets.

Let $E = \bigcup_{k=1}^{\infty} E_k$ where $\{E_k\}$ is a countable collection of disjoint measurable set.

Claim: E is measurable

Let A be any set

consider $F_n = \bigcup_{k=1}^{n} E_k$

Then F_n is measurable set.

$$\mu^*(A) = \mu^*(A \cap F_n) + \mu^*(A \cap F_n^c)$$

$$= \mu^*\left[A\left(\bigcup_{k=1}^{n} E_k\right)\right] + \mu^*(A \cap F_n^c)$$

$$= \sum_{k=1}^{n} \mu^*(A \cap E_k) + \mu^*(A \cap F_n^c)$$

Since $F_n = \bigcup_{k=1}^{n} \subset E$, $F_n^c \supset E^c$

$$\therefore \mu^*(A) \geq \sum_{k=1}^{n} \mu^*(A \cap E_k) + \mu^*(A \cap E^c) \ \forall n$$

$$\therefore \mu^*(A) \geq \sum_{k=1}^{\infty} \mu^*(A \cap E_k) + \mu^*(A \cap E^c).$$

$$\mu^*(A) \geq \mu^*\left(A \cap \bigcup_{k=1}^{\infty} E_k\right) + \mu^*(A \cap E^c)$$

$$= \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

(ie) $\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E^c)$

Hence E is measurable.

Proposition : 8

Let μ^* be an outer measure on 2^x then the collection M of all measurable set w.r.to μ^* is a σ-algebra. If $\bar{\mu}$ is the restriction of μ^* on M Then $(X, M, \bar{\mu})$ is a complete measure space

Proof:

Since $\mu^*(\phi) = 0$, $\phi \in M$

If $E \in M$, then $E^c \in M$

By countable union of measurable set is measurable.

$\therefore$ M is a σ-algebra.

It is clear that $M \subseteq 2^x$

Given that $\bar{\mu} = \mu^*/M$

Claim: $\bar{\mu}$ is measure on M.

Since $\phi \in M$, $\bar{M}(\phi) = 0 = \mu^*(\phi)$

(ie) $\bar{\mu}(\phi) = 0$ —— (1)

Let $\{E_k\}_{k=1}^{\infty}$ be a countable collection of disjoint measurable sets of M

Take $E = \bigcup_{k=1}^{\infty} E_k$, then $E \in M$.

$\therefore \bar{\mu}(E) = \mu^*(E) \leq \sum_{k=1}^{\infty} \mu^*(E_k)$

(ie) $\bar{\mu}(E) \leq \sum_{k=1}^{\infty} \mu^*(E_k) = \sum_{k=1}^{\infty} \bar{\mu}(E_k)$

$\therefore \bar{\mu}(E) \leq \sum_{k=1}^{\infty} \bar{\mu}(E_k)$ —— (2)

clearly $\bigcup_{k=1}^{\infty} E_k \supseteq \bigcup_{k=1}^{n} E_k$ for any n.

Since μ^* is monotone,

$$\mu^*\left(\bigcup_{k=1}^{\infty} E_k\right) \geq \bar{\mu}^*\left(\bigcup_{k=1}^{n} E_k\right)$$

$$= \sum_{k=1}^{n} \mu^*(E_k) \ \forall\ n$$

(ie) $\bar{\mu}(E) \geq \sum_{k=1}^{n} \bar{\mu}(E_k)$ for any n.

$$\bar{\mu}(E) \geq \sum_{k=1}^{\infty} \bar{\mu}(E_k) \text{ ——— (3)}$$

From (1) & (2)

$$\bar{\mu}(E) = \sum_{k=1}^{\infty} \bar{\mu}(E_k)$$

$\therefore \bar{\mu}$ is countably additive

$\therefore \bar{\mu}$ is measure on E.

Claim: $(X, M, \bar{\mu})$ is a complete measure space

Let $E \in M$, with $\bar{\mu}(E) = 0$

Let $E_1 \subset E$

Since μ^* is monotone,

$$\mu^*(E_1) \leq \mu^*(E) = \bar{\mu}(E) = 0$$

(ie) $\mu^*(E_1) = 0$

$E_1 \in M$

(ie) Every subset of measure zero is measure.

$\therefore (X, M, \bar{\mu})$ is complete measure space

The construction of outer measure

Proposition: 9

Let S be a collection of subsets of a set X. Let $\mu: S \to [0, \infty]$ be a set function define μ^* as $\mu^*(\phi) = 0$ and for $E \subseteq X$, $E \neq \phi$.

$\mu^*(E) = \inf \left\{ \sum_{k=1}^{\infty} \mu(E_k) \mid E \subset \bigcup_{n=1}^{\infty} E_k \right\}$ where is taken over all countable collection $\{E_k\}_{k=1}^{\infty}$ of sets in that over E. Then $\mu^* : 2^x \longrightarrow [0, \infty]$ is an outer measure, this outer measure μ^* is called outer measure included by μ.

Proof:

Given that $\mu^*(\phi) = 0$

T.P: μ^* is countably monotone.

Let $E \subset 2^x$, $E \subset \bigcup_{k=1}^{\infty} E_k$, $E_k \in S$.

suppose $\mu^*(E_k) = \infty$ for some k.

Then trivially $\mu^*(E) \leq \sum_{k=1}^{\infty} \mu^*(E_k)$

suppose $\mu^*(E_k) < \infty \ \forall k$.

let $\varepsilon > 0$ be given.

Then for each k, there is collection $\{E_k\}_{k=1}^{\infty}$ of sets in S that covers E_k such that

$\mu^*(E_k) + \frac{\varepsilon}{2^k} > \sum_{k=1}^{\infty} \mu(E_{i,k})$.

Now $\{E_{i,k}\}_{1 \leq i, k \leq \infty}$ is a countable collection of sets in S. that covers E.

(ii) $E \subseteq \bigcup_{1 \leq i, k \leq \infty} \mu(E_{i,k}) = \sum_{k=1}^{\infty} \left[\sum_{i=1}^{\infty} \mu(E_{i,k}) \right]$

$< \sum_{k=1}^{\infty} \left[\mu^*(E_k) + \frac{\varepsilon}{2^k} \right]$

$= \sum_{k=1}^{\infty} \mu^*(E_k) + \varepsilon$.

Since $\varepsilon > 0$ is arbitrary

$\mu^*(E) \leq \sum_{k=1}^{\infty} \mu^*(E_k)$

$\therefore \mu^*$ is countably monotone.

μ^* is an outer measure.

Note:

Thus outer measure μ^* is the outer measure induced by μ.

Let S be a collection of subsets of x $\mu: S \to [0, \infty]$ a set function on μ^* the outer measure induced by μ. Let M be the collection of all μ^* measurable sets. The M is a σ-algebra Let $\bar{\mu}$ is a measure and $(X, M, \bar{\mu})$ is a complete measure space μ is called Carathodory measure induced by $\mu, \mu^*, 2^X \longrightarrow [0, \infty]$.

$\mu: S \to [0, \infty]$. (a general set function)

$\mu: M \to [0, \infty]$ (The induced Carathod measure)

Remark:

For a collection S of subset of x, S_σ - denote these sets that are countable union of sets in S. $S_{\sigma s}$ denote the sets that are countable intersection of sets is S_σ.

Proposition: 10

Let $\mu: S \to [0, \infty]$ be a set function defined on a callections of subsets of a set x and $\mu: M \to [0, \infty]$ the Carathodory measure induced by μ. Let E be a subset of x for which $\mu^*(E) < \infty$ then there is a subset A of x for which $A \in S_{\sigma s}$, $E \subseteq A$ and $\mu^*(E) = \mu^*(A)$. Furthermore if E and each sets in S measurable w.r.to then So it A and $\bar{\mu}(A \sim E) = 0$.

Proof

Let $\varepsilon > 0$

Since $\mu^*(E) < \infty$ there is a cover of E by a collection $\{E_k\}_{k=1}^{\infty}$ of sets in S for which

$$\sum_{k=1}^{\infty} \mu(E_k) < \mu^*(E) + \varepsilon \quad \text{——— ①}$$

Define $A_\varepsilon = \bigcup_{k=1}^{n} E_k$, then $A_\varepsilon \in S_\sigma$, $E \subseteq A$

Now,

$$\mu^*(A_\varepsilon) \leq \sum_{k=1}^{n} \mu(E_k) < \mu^*(E) + \varepsilon$$

Define $A = \bigcap_{k=1}^{\infty} A_{1/k}$ then $A \in S_{\sigma\delta}$, $E \subseteq A$.

$$\therefore \mu^*(E) \leq \mu^*(A) \leq \mu^*(A_{1/k})$$
$$\leq \mu^*(E) + \frac{1}{k} \quad \forall k$$
$$\mu^*(E) \leq \mu^*(A) \leq \mu^*(E)$$
$$\therefore \mu^*(E) = \mu^*(A) \quad \text{——— ②}$$

Assume that E is μ^* measurable and each set in S is μ^* measurable.

Then $A = \bigcap_{k=1}^{\infty} A_{1/k}$ is μ^* measurable.

Since $\bar{\mu} = \mu^*$ (μ is a measure and $E \subseteq A$, by exision property).

$$\bar{\mu}(A \sim E) = \bar{\mu}(A) - \bar{\mu}(E)$$
$$= \mu^*(A) - \mu^*(E)$$
$$= 0 \quad [\because \mu^*(E) \leq \mu^*(A)].$$
$$\bar{\mu}(A \sim E) = 0$$